Kinetic Equations

Text of the Exercises

– 03.03.2021 –

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Exercise 1

Let $T \geqslant 0$ be a positive real number and $b \in C^1\left([0,T] \times \mathbb{R}^d; \mathbb{R}^d\right)$ be a bounded vector field. Let $X \in C^1\left([0,T] \times [0,T] \times \mathbb{R}^d; \mathbb{R}^d\right)$ be the flow associated to b, i.e., the unique differentiable solution to

$$\begin{cases} \partial_s X(s,t,x) = b(s, X(s,t,x)), & \forall (s,t,x) \in [0,T] \times [0,T] \times \mathbb{R}^d, \\ X(t,t,x) = x, & \forall (t,x) \in [0,T] \times \mathbb{R}^d \end{cases} \tag{1}$$

a Prove that X satisfies the semigroup property, i.e.,

$$X(r, s, X(s,t,x)) = X(r,t,x), \qquad \forall\, r,s,t \in [0,T],\ \forall x \in \mathbb{R}^d. \tag{2}$$

b Use point **a** to prove that for any $s,t \in [0,T]$ the map $X(s,t,\cdot) \in C^1\left(\mathbb{R}^d; \mathbb{R}^d\right)$ is a C^1 diffeomorphism, i.e. it is invertible with its inverse in $C^1\left(\mathbb{R}^d; \mathbb{R}^d\right)$.

Exercise 2

Let $T \geqslant 0$ be a positive real number, $b \in C^1\left([0,T] \times \mathbb{R}^d; \mathbb{R}^d\right)$ be a bounded vector field and $X \in C^1\left([0,T] \times [0,T] \times \mathbb{R}^d; \mathbb{R}^d\right)$ be again the flow associated to b. Define the *Jacobian* $J \in C\left([0,T] \times [0,T] \times \mathbb{R}^d; \mathbb{R}\right)$ as we did in class as

$$J(s,t,x) := \det(\nabla_x X)(s,t,x). \tag{3}$$

From classical results in the theory of ordinary differential equations, $\partial_s J$ exists and is in $C\left([0,T] \times [0,T] \times \mathbb{R}^d; \mathbb{R}\right)$.

Show that $J(s,t,x) > 0$ for all $(s,t,x) \in [0,T] \times [0,T] \times \mathbb{R}^d$ and that J solves

$$\begin{cases} (\partial_s J)(s,t,x) = (\operatorname{div}_x b)(s, X(s,t,x))\, J(s,t,x), & \forall (s,t,x) \in [0,T] \times [0,T] \times \mathbb{R}^d, \\ J(t,t,x) = 1, & \forall (t,x) \in [0,T] \times \mathbb{R}^d. \end{cases} \tag{4}$$

Prove moreover that J satisfies

$$\partial_t J(s,t,x) + \operatorname{div}_x \left(b(t,x)\, J(s,t,x)\right) = 0. \tag{5}$$

Hint: You can assume without proof that $\partial_s \nabla_x X$ exists and it is in $C\left([0,T] \times [0,T] \times \mathbb{R}^d; \mathbb{R}\right)$, and is equal to $\nabla_x \partial_s X$. For a proof of this result, one can look at Theorem 2.10 in the book ***Ordinary Differential Equations and Dynamical Systems*** *from Gerald Teschl, available online for free.*

Exercise 3

Let $T \geqslant 0$ be a positive real number, $b \in C^2\left([0,T] \times \mathbb{R}^d; \mathbb{R}^d\right)$ be a bounded vector-field. Assume that $u_0 \in C^1\left(\mathbb{R}^d\right)$ and that $f \in C^1\left([0,T] \times \mathbb{R}^d\right)$.

Prove that there exists a unique solution $u \in C^1\left([0,T] \times \mathbb{R}^d\right)$ for the inhomogeneous transport equation

$$\begin{cases} \partial_t u\left(t,x\right) + \operatorname{div}_x\left(b\left(t,x\right) u\left(t,x\right)\right) = f\left(t,x\right), & \forall\left(t,x\right) \in [0,T] \times \mathbb{R}^d, \\ u\left(0,x\right) = u_0\left(x\right), & \forall\ x \in \mathbb{R}^d. \end{cases} \tag{6}$$